Sydney Technical High School

Mathematics Department

Trial HSC – Mathematics 2 Unit

August 2016

General Instructions

- Reading time – 5 minutes.
- Working time – 180 minutes.
- Approved calculators may be used.
- Write using blue or black pen.
- A BOSTES reference sheet is provided at the back of this paper. You may tear it off.
- In Question 11-16, show relevant mathematical reasoning and/or calculations.
- Begin each question on a new page of the answer booklet.
- Marks shown are a guide and may need to be adjusted.
- Full marks may not be awarded for careless work or illegible writing.

NAME: ______________________

TEACHER: ______________________

Total marks – 100

SECTION 1

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes.

SECTION 2

90 marks

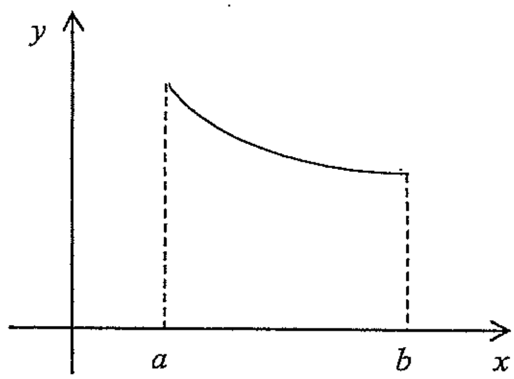
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes.

Section 1 **(10 marks)**

1. For what values of k does the equation $x^2 - 6x - 3k = 0$ have real roots?

A) $k \geq -3$ B) $k \leq -3$ C) $k \geq 3$ D) $k \leq 3$

2. For the function $y = f(x)$, $a < x < b$ graphed below:



Which of the following is true?

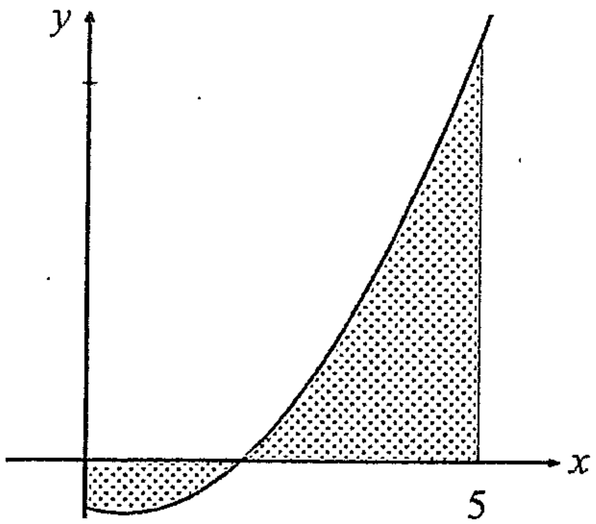
A) $f'(x) > 0$ and $f''(x) > 0$

B) $f'(x) > 0$ and $f''(x) < 0$

C) $f'(x) < 0$ and $f''(x) > 0$

D) $f'(x) < 0$ and $f''(x) < 0$

3. Which expression will give the area of the shaded region bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 0$ and $x = 5$?



A) $A = \left| \int_0^1 (x^2 - x - 2)dx \right| + \int_1^5 (x^2 - x - 2)dx$

B) $A = \int_0^1 (x^2 - x - 2)dx + \left| \int_1^5 (x^2 - x - 2)dx \right|$

C) $A = \left| \int_0^2 (x^2 - x - 2)dx \right| + \int_2^5 (x^2 - x - 2)dx$

D) $A = \int_0^2 (x^2 - x - 2)dx + \left| \int_2^5 (x^2 - x - 2)dx \right|$

4. What are the coordinates of the focus of the parabola $4y = x^2 - 8$?

A) (0, -8) B) (0, -7) C) (0, -2) D) (0, -1)

5. What are the domain and range of the function $f(x) = \sqrt{4 - x^2}$?

A) Domain: $-2 \leq x \leq 2,$ Range: $0 \leq y \leq 2$

B) Domain: $-2 \leq x \leq 2,$ Range: $-2 \leq y \leq 2$

C) Domain: $0 \leq x \leq 2,$ Range: $-4 \leq y \leq 4$

D) Domain: $0 \leq x \leq 2,$ Range: $0 \leq y \leq 4$

6. When the curve $y = e^x$ is rotated about the $x - axis$ between $x = -2$ and $x = 2$, the volume of the solid generated is given by:

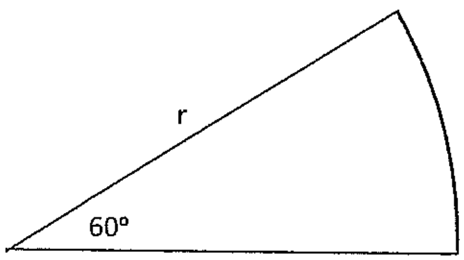
A) $\pi \int_{-2}^{2} e^x dx$

B) $2\pi \int_{0}^{2} e^{x^2} dx$

C) $\pi \int_{-2}^{2} e^{x^2} dx$

D) $\pi \int_{-2}^{2} e^{2x} dx$

7. The sector below has an area of 10π square units.



What is the value of r?

A) $\sqrt{60}$

B) $\pi\sqrt{60}$

C) $\sqrt{\frac{\pi}{3}}$

D) $\sqrt{\frac{1}{3}}$

8. An infinite geometric series has a first term of 8 and a limiting sum of 12. What is the common ratio?

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$

9. If $\int_0^a 4 - 2x \, dx = 4$, find the value of a.

A) $a = -2$ B) $a = 0$ C) $a = 4$ D) $a = 2$

10. What is the greatest value taken by the function $f(x) = 4 - 2cosx$ for $x \geq 0$?

A) 2 B) 4 C) 6 D) 8

Section 2 (90 marks)

Question 11 (15 marks) Marks

a) Find $\sqrt[3]{9.8^2}$ correct to 2 decimal places 1

b) Factorise fully $ax + 3ay - x - 3y$ 1

c) Solve for a and d: 1

$$a + 9d = 20$$

$$2a + 9d = 12$$

d) Express $\frac{2}{5+\sqrt{3}}$ with a rational denominator 1

e) Solve $|3x - 1| = 5$ 2

f) Solve the following equation: 2

$$\log_2 x + \log_2(x + 7) = 3$$

g) Solve $\cos x = \frac{-1}{2}$ for $0 \leq x \leq 2\pi$ 2

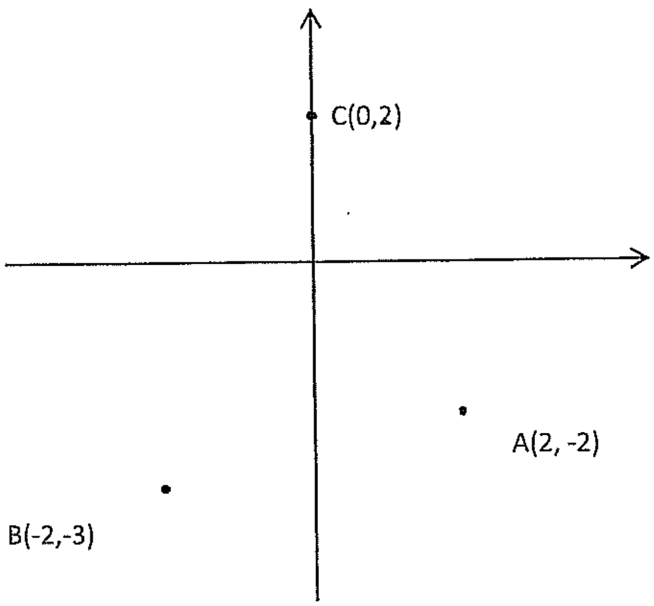
h) Find the primitive of $x^2\sqrt{x}$ 2

i) Differentiate $\frac{3}{(2x+1)^2}$ 2

j) Find $\int_0^1 e^{2x}dx$ 1

Question 12 (15 marks) Marks

a) On the diagram below, A (2, -2) B (-2, -3) and C (0, 2) are the vertices of a triangle ABC. Copy this diagram into your answer booklet.



i) Find the gradient of AC — 1

ii) Find the angle of inclination that AC makes with the positive direction of the x *axis*, to the nearest degree. — 1

iii) Show that the equation of AC is $2x + y - 2 = 0$ — 1

iv) Calculate the perpendicular distance of B from the line AC — 2

v) Find the area of Δ ABC — 2

vi) Find the coordinates of D such that ABCD is a parallelogram. — 1

b) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{3x}$ — 2

c) In Δ ABC, AB = 2cm, $\angle$ ABC = 105° and $\angle$ BCA = 30°. Find the length of BC correct to 1 d.p. — 2

d) Max is saving to buy a new car. He needs \$12700. In the first month he saves \$25, in the second \$40 followed by \$55 in the next. If he continues to increase the amount he saves by \$15 each month, how many months will it take him to save for the car? — 3

Question 13 (15 marks) Marks

a) Differentiate:

i) $x\tan 2x$ 2

ii) $e^{\sin x} + \frac{1}{x}$ 2

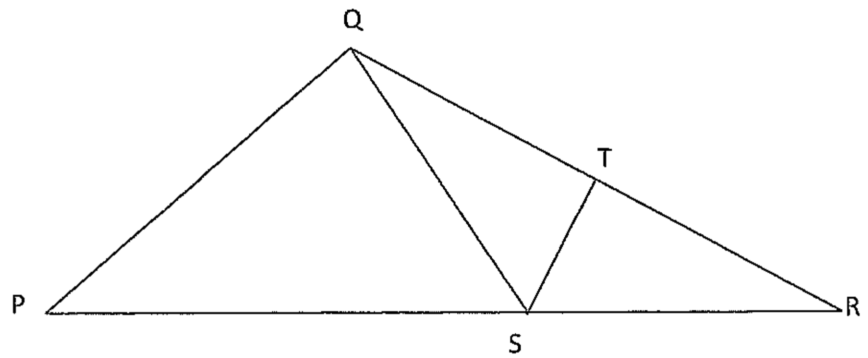
iii) $\frac{3x-7}{3+2x}$ 2

b) Find

i) $\int (5x-1)^9\, dx$ 2

ii) $\int \sin\frac{3x}{4}\, dx$ 2

c)



In Δ PQR, point T lies on side QR and point S lies on side PR such that QT = TR, QS = QP and ST $\perp$ QT.

i) Copy the diagram into your answer booklet showing all given information. 1

ii) Prove that Δ QTS $\equiv$ Δ RTS 2

iii) Prove that $\angle$ QPS = 2 $\angle$ TQS 2

Question 14 (15 marks) Marks

a) Consider the curve

$$f(x) = -\frac{1}{3}x^3 - x^2 + 3x + 1$$

i) Find the coordinates of any stationary points and determine their nature. 3

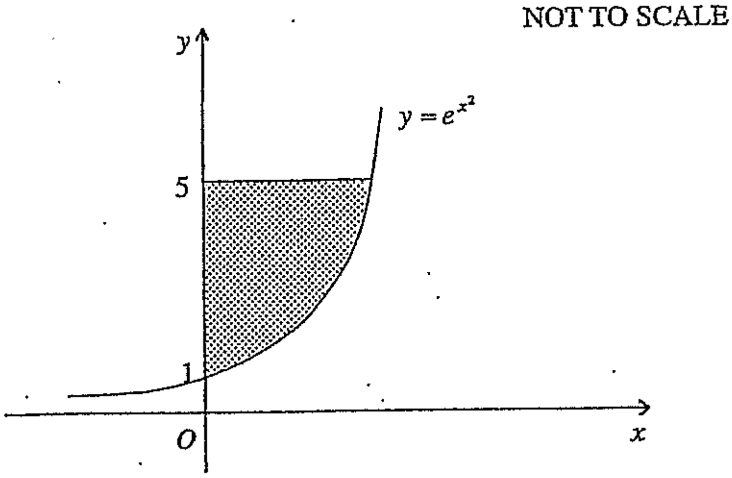
ii) Find any point(s) of inflexion 2

iii) Sketch the curve in the domain, $-6 \leq x \leq 3$ 2

iv) What is the maximum value of $f(x)$ in the given domain? 1

b) Simplify $\frac{1-\sin^2 x}{\cot x}$ 2

c)



The shaded region bounded by the graph $y = e^{x^2}$, the line $y = 5$ and the y axis is rotated about the y – axis to form a solid revolution.

i) Show that the volume of the solid is given by 1

$$V = \pi \int_1^5 \log_e y \, dy$$

Marks

ii) Copy and complete the following table into your writing booklet. 1

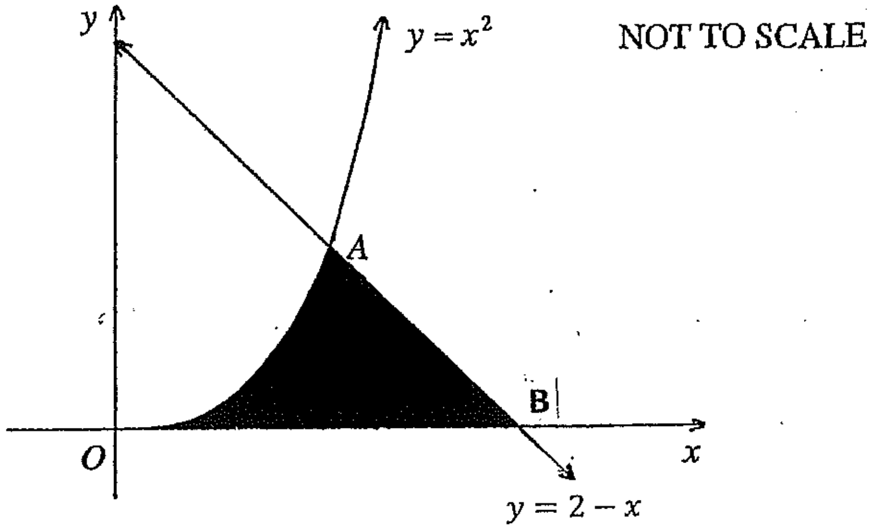
Give your answer correct to 3 decimal places.

y	1	2	3	4	5
$log_e y$	0	0.693	1.099		1.609

iii) Use Simpson's Rule with five function values to approximate the volume of the solid of revolution V_y, correct to three decimal places. 3

Question 15 (15 marks) Marks

a)



The shaded region OAB is bounded by the parabola $y = x^2$, the line $y = 2 - x$ and the $x - axis$.

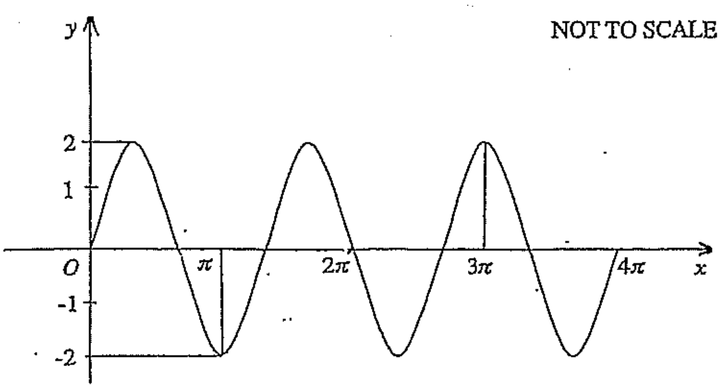
i) Find the x coordinates of A and B. 2

ii) Show that the exact area of the shaded region OAB is given by $\frac{5}{6}$ square units. 2

b) i) Show that $\frac{d}{dx}(xe^x) = e^x + xe^x$ 1

ii) Find $\int xe^x dx$ 2

c) Find the trigonometric equation for the graph below: 2



Question 15 (cont) Marks

d) Mr Egan borrows \$P from a bank to fund his house extensions. The term of the loan is 20 years with an annual interest rate of 9%. At the end of each month, interest is calculated on the balance owing and added to the balance owing. Mr Egan repays the loan in equal monthly instalments of \$1050.

i) Write an expression for the amount, A_1, Mr Egan owes at the end of the first month — 1

ii) Show that at the end of n months, the amount owing, A_n, is given by: — 3

$A_n = P(1.0075)^n - 140000(1.0075)^n + 140000$

iii) If the loan is repaid at the end of 20 years, calculate the amount Mr Egan originally borrowed, correct to the nearest dollar. — 2

Question 16 (15 marks)

a) Find $\int 2^x \, dx$ — 1

b) Let $\propto$ *and* β be the solutions of $x^2 + 5x + 3 = 0$. Find:

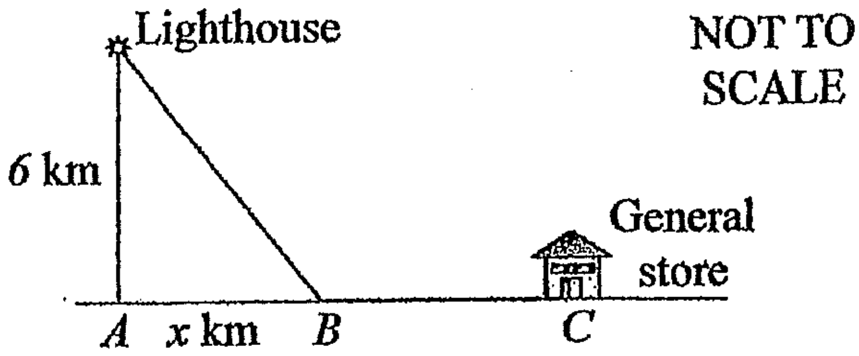
i) $\frac{1}{\propto} + \frac{1}{\beta}$ — 2

ii) A quadratic equation whose roots are $\frac{1}{\propto}$ and $\frac{1}{\beta}$ — 2

c) Evaluate $\int_0^2 \frac{6x}{x^2+2} \, dx$ — 3

Question 16 (cont) Marks

d)



The water's edge is a straight line ABC which runs east-west. A lighthouse is 6km from the shore on a rocky outcrop, due north of A.

10km due east of A is a general store. To get to the general store as quickly as possible the lighthouse keeper rows to a point B, $x$km from A, and then jogs to the general store.

The lighthouse keeper's rowing speed is 6km/h and his jogging speed is 10km/h.

i) Show that it takes the lighthouse keeper $\frac{\sqrt{36+x^2}}{6}$ hours to row from the lighthouse to B. 2

ii) Show that the total time taken for the lighthouse keeper to reach the general store is given by 1

$$T = \frac{\sqrt{36+x^2}}{6} + \frac{10-x}{10} \text{ hours}$$

iii) Hence, show that when $x = 4\frac{1}{2}$km, the time it takes the lighthouse keeper to travel from the lighthouse to the general store is a minimum (you may assume it is a minimum – no testing required) 3

iv) Find the quickest time it takes the lighthouse keeper to go to the general store from the lighthouse. (You may leave your answer in hours). 1

Student Name: ______________ Teacher Name: ______________

2016 2 Unit Trial Solutions

Section 1

1. A 2. C 3. C 4. D 5. A 6. D 7. A
8. C 9. D 10. C

Section 2

Question 11

a) 4.58

b) $a(x+3y)-(x+3y)$
$(a-1)(x+3y)$

c) $a=-8$
$d=\frac{28}{9}$

d) $\frac{2}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}}$
$=\frac{10-2\sqrt{3}}{22}$
$=\frac{5-\sqrt{3}}{11}$

e) $3x-1=5$ $\quad$ $3x-1=-5$
$x=2$ (1) $\quad$ $x=\frac{-4}{3}$ (1)

f) $\log_2 x(x+7)=3$
$x(x+7)=8$
$x^2+7x-8=0$
$(x-1)(x+8)=0$
$x=1$ or ~~-8~~
$x=1$ as $x>0$
(1) (1)

g) $\cos x=\frac{-1}{2}$
working angle $\frac{\pi}{3}$

S ✓	A
T ✓	C

$\therefore x=\pi-\frac{\pi}{3}, \pi+\frac{\pi}{3}$
$x=\frac{2\pi}{3}, \frac{4\pi}{3}$ (1) (1)

h) $x^2\sqrt{x}$
$\int x^{\frac{5}{2}}dx$ (1)
$=\frac{2}{7}x^{\frac{7}{2}}+C$ (1)

i) $\frac{d}{dx}\frac{3}{(2x+1)^2}$
$\frac{d}{dx}3\times(2x+1)^{-2}$
$=-12(2x+1)^{-3}$ (1) (1)

j) $\int_0^1 e^{2x}dx$
$\left[\frac{1}{2}e^{2x}\right]_0^1$
$\frac{1}{2}(e^2-1)$

Student Name: ______________ Teacher Name: ______________

Question 12

a) (i) $m_{AC}=\frac{2--2}{0-2}$
$=\frac{4}{-2}$
$m_{AC}=-2$

(ii) $m=\tan\theta$
$-2=\tan\theta$
$\theta=117^\circ$

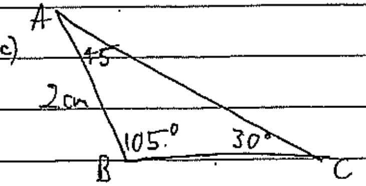
(iii) $y-2=-2(x-0)$
$y-2=-2x$
$2x+y-2=0$

(iv) $d=\frac{|2\times-2+1\times-3+-2|}{\sqrt{2^2+1^2}}$ (1)
$=\frac{9}{\sqrt{5}}$ units (1)

(v) $A=\frac{1}{2}\times AC\times\frac{9}{\sqrt{5}}$
$=\frac{9}{2\sqrt{5}}\times\sqrt{2^2+(-2-2)^2}$
$=\frac{9}{2\sqrt{5}}\times\sqrt{20}$ (1)
$=\frac{9}{2\sqrt{5}}\times2\sqrt{5}$
$=9$ units^2 (1)

(vi) $D(4,3)$

b) $\frac{2}{3}\lim_{x\to0}\frac{\sin 2x}{2x}$ (1)
$=\frac{2}{3}$ (1)

c)


$\frac{BC}{\sin 45}=\frac{2}{\sin 30}$ (1)
$BC=\frac{2\sin 45}{\sin 30}$
$BC=2.8$ cm (1)

d) $12700=25+40+55+\dots$ (1)
$S_n=12700$ $\quad a,\ d=15,\ n=?$
$12700=\frac{n}{2}\{2\times25+(n-1)15\}$
$25400=50n+15n^2-15n$
$15n^2+35n-25400=0$ (1)
$3n^2+7n-5080=0$
$n=\frac{-7\pm\sqrt{49+12\times5080}}{6}$
(1) $n=40$ $(n>0)$

Student Name: ______ Teacher Name: ______

Question 13

a)(i) $\frac{d}{dx}(x\tan 2x)$

$= \tan 2x + x \times 2\sec^2 2x$ ①

$= \tan 2x + 2x\sec^2 2x$ ①

(ii) $\frac{d}{dx}\left(e^{\sin x} + x^{-1}\right)$

$= \cos x\, e^{\sin x} - x^{-2}$ ① ①

(iii) $\frac{d}{dx}\left(\frac{3x-7}{3+2x}\right)$

$\frac{(3+2x)3-(3x-7)2}{(3+2x)^2}$ ①

$= \frac{23}{(3+2x)^2}$ ①

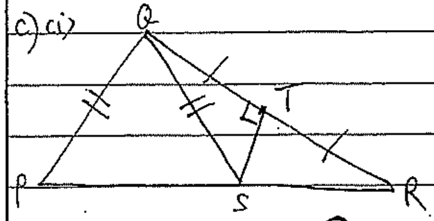
b)(i) $\int (5x-1)^9\,dx$

$= \frac{(5x-1)^{10}}{50}$ ②

(ii) $\int \sin\frac{3x}{4}\,dx$

$= -\frac{4}{3}\cos\frac{3x}{4} + C$ ②

c)(i)



①

(ii) In Δ's QTS and RTS,

ST is common

QT = TR given

$\angle QTS = \angle RTS$ (straight angle ST ⊥ QR) ①

$\therefore \Delta QTS \equiv \Delta RTS$ (SAS) ①

(iii) Let $\angle TQS = \theta$

$\therefore \angle TRS = \theta$ (corresponding angles in congruent Δ's)

$\therefore \angle QST = \angle RST = 90 - \theta$ (angle sum of Δ's) ①

$\angle QSP = 180 - 2(90-\theta)$ (straight angle)

$= 2\theta$

$\angle QPS = 2\theta$ (equal angles opposite equal sides of a triangle)

$\therefore \angle QPS = 2\angle TQS$ ①

Student Name: ______ Teacher Name: ______

Question 14

a) $f(x) = -\frac{1}{3}x^3 - x^2 + 3x + 1$

$f'(x) = -x^2 - 2x + 3 = 0$

$x^2 + 2x - 3 = 0$ ①

$(x-1)(x+3) = 0$

$x = 1$ or -3

$f''(x) = -2x - 2$ ①

$f''(1) = -4 < 0 \therefore (1, 2\frac{2}{3})$ max.

$f''(-3) = 4 > 0 \therefore (-3, -8)$ min. ①

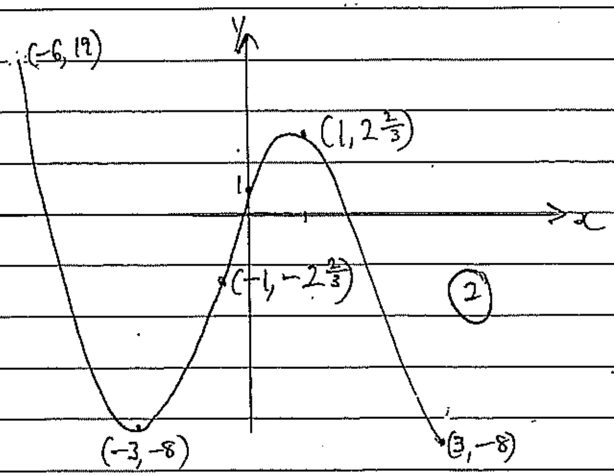
(ii) $f''(x) = 0$ for pts. of inflexion

$-2x - 2 = 0$ ①

$x = -1$

$(-1, -2\frac{2}{3})$ a non horizontal inflexion ① as $f'(-1) \neq 0$. No testing required.

(iii)



②

(iv) 19 ①

b) $\frac{\cos^2 x}{\frac{\cos x}{\sin x}}$ ①

$= \frac{\cos^2 x}{1} \times \frac{\sin x}{\cos x}$

$= \sin x \cos x$ ①

c)(i) $V = \pi \int x^2\,dy$

If $y = e^{x^2}$ then $\log_e y = x^2$

$\therefore V = \pi \int_1^5 \log_e y\,dy$ ①

Student Name: ____________ Teacher Name: ____________

(ii)

y	1	2	3	4	5
$\log_e y$	0	0·693	1·099	1·386	1·609

(1)

(iii) $V_y \doteqdot \frac{1}{3}\{0 + 1\cdot609 + 4(0\cdot693 + 1\cdot386) + 2\times1\cdot099\}\times\pi$ (2)

$= 12\cdot695$ (1)

Student Name: ____________ Teacher Name: ____________

Question 15

a)(i) $x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1$ or ~~-2~~

At A $x = 1$, (>0) (1)

$y = 2 - x$

$0 = 2 - x$

$\therefore x = 2$ at B (1)

(ii) $A = \int_0^1 x^2\,dx + \int_1^2 2 - x\,dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$ (1)

$= \frac{1}{3} + (4-2) - (2 - \frac{1}{2})$

$= \frac{1}{3} + 2 - 2 + \frac{1}{2}$

$= \frac{5}{6}$ units^2 (1)

b) (i) $\frac{d}{dx}(xe^x)$

$= xe^x + e^x \times 1$

$= e^x + xe^x$ (1)

(ii) $\frac{d}{dx}(xe^x) = e^x + xe^x$

$\frac{d}{dx}(xe^x) - e^x = xe^x$ (1)

$\int \frac{d}{dx}(xe^x) - e^x\,dx = \int xe^x\,dx$

$xe^x - e^x + C = \int xe^x\,dx$ (1)

c) Amplitude 2

Period $= \frac{4\pi}{3}$

$\therefore T = \frac{2\pi}{n} = \frac{4\pi}{3} \therefore n = \frac{3}{2}$

Curve is of the form $y = A\sin nx$

$\therefore y = 2\sin\frac{3x}{2}$

(1) (1)

d)(i) $A_1 = P\times\left(1+\frac{\frac{9}{12}}{100}\right) - 1050$

$A_1 = P\times(1.0075) - 1050$ (1)

(ii) $A_2 = A_1\times 1.0075 - 1050$

$= [P\times 1.0075 - 1050]\times 1.0075 - 1050$

$= P\times 1.0075^2 - 1050(1+1.0075)$

$\vdots$

$A_n = P\times 1.0075^n - 1050(1+1.0075+\dots 1.0075^{n-1})$ (1)

$a=1,\ r=1.0075,\ n=n$

$= P\times 1.0075^n - 1050\times\frac{1\times(1.0075^n-1)}{1.0075-1}$ (1)

$= P\times 1.0075^n - 140000(1.0075^n - 1)$

$= P\times 1.0075^n - 140000\times 1.0075^n + 140000$ (1)

(iii) At 20 years, $n=240$, $A_n=0$, solve P

$0 = P\times 1.0075^{240} - 140000\times 1.0075^{240} + 140000$

$P = \$116702$ (2)

Question 16

a) $\int 2^x\,dx$

$\frac{1}{\log_e 2}2^x + C$ (1)

b)(i) $\alpha+\beta = -5$

$\alpha\beta = 3$

$\frac{1}{\alpha}+\frac{1}{\beta}$

$= \frac{\alpha+\beta}{\alpha\beta}$ (1)

$= \frac{-5}{3}$ (1)

(ii) $x^2 - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x + \frac{1}{\alpha\beta}$

$x^2 - \left(\frac{-5}{3}\right)x + \frac{1}{3} =$

$3x^2+5x+1=0$

or

$x^2+\frac{5}{3}x+\frac{1}{3}=0$

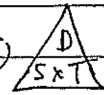
c) $\int_0^2 \frac{6x}{x^2+2}\,dx$

$3\int_0^2 \frac{2x}{x^2+2}\,dx$ (1)

$= 3\left[\log_e(x^2+2)\right]_0^2$ (1)

$= 3\left[\log_e 6 - \log_e 2\right]$

$= 3\log_e 3$ (1)

d)(i) 

$T = \frac{D}{S}$

distance from B to lighthou

$= \sqrt{x^2+36}$ km (1)

$\therefore T = \frac{\sqrt{x^2+36}}{6}$ hours (1)

(ii) Running:

$T = \frac{\text{distance BC}}{\text{running speed}}$

$= \frac{10-x}{10}$

$\therefore$ Total time T

$= \frac{\sqrt{x^2+36}}{6} + \frac{10-x}{10}$

(iii) $\frac{dT}{dx} = \frac{1}{6}\times\frac{1}{2}(x^2+36)^{-\frac{1}{2}}\times 2x - \frac{1}{10}$

$= \frac{x}{6\sqrt{x^2+36}} - \frac{1}{10} = 0$ for a minimum.

(1)

Student Name: ______________________ Teacher Name: ______________________

$$\frac{x}{6\sqrt{x^2+36}} = \frac{1}{10}$$

$$10x = 6\sqrt{x^2+36}$$

$$\frac{5x}{3} = \sqrt{x^2+36}$$

$$\frac{25x^2}{9} = x^2+36 \quad (1)$$

$$25x^2 = 9x^2 + 324$$

$$16x^2 = 324$$

$$x^2 = 20.25$$

$$x = 4.5\text{ Km} \quad (1)$$

(iv) Sub $x = 4.5$ into expression for T.

$$T = \frac{\sqrt{4.5^2+36}}{6} + \frac{10-4.5}{10}$$

$$= 1.8 \text{ hours} \quad (1)$$